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Solution to exercises.

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Solutions to Exercises

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2.9 Exercises

Exercise 2.1

a. ETS(A,A_d,N)

$$\begin{aligned}
y_t &= \ell_{t-1} + \phi b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta\varepsilon_t
\end{aligned}$$

b.

$$\begin{aligned}
\boldsymbol{x}_t &= \begin{bmatrix}\ell_t & b_t\end{bmatrix}' \\
y_t &= \begin{bmatrix}1 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} + \varepsilon_t \\
\boldsymbol{x}_t &= \begin{bmatrix}1 & \phi \\ 0 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} + \begin{bmatrix}\alpha \\ \beta\end{bmatrix}\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} & \qquad \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= 1 \\
\boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi \\ 0 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} & \qquad \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}\alpha \\ \beta\end{bmatrix}
\end{aligned}$$

c. ETS(A,A,A)

$$\begin{aligned}
y_t &= \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t \\
s_t &= s_{t-m} + \gamma\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{x}_t &= \begin{bmatrix}\ell_t & b_t & s_t\end{bmatrix}' \\
y_t &= \begin{bmatrix}1 & 1 & 1\end{bmatrix}\boldsymbol{x}_{t-1} + \varepsilon_t \\
\boldsymbol{x}_t &= \begin{bmatrix}1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1} + \begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & 1 & 1\end{bmatrix}\boldsymbol{x}_{t-1} & \qquad \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= 1 \\
\boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1} & \qquad \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}
\end{aligned}$$

d. ETS(M,$\mathrm{A_d}$,N)

$$\begin{aligned}y_t &= (\ell_{t-1} + \phi b_{t-1})(1+\varepsilon_t)\\ \ell_t &= (\ell_{t-1} + \phi b_{t-1})(1+\alpha\varepsilon_t)\\ b_t &= \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t\end{aligned}$$

$$\begin{aligned}\boldsymbol{x}_t &= \begin{bmatrix}\ell_t & b_t\end{bmatrix}'\\ y_t &= \begin{bmatrix}1 & \phi\end{bmatrix}\boldsymbol{x}_{t-1}(1+\varepsilon_t)\\ \boldsymbol{x}_t &= \begin{bmatrix}1 & \phi\\ 0 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} + \begin{bmatrix}1 & 1\end{bmatrix}\boldsymbol{x}_{t-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\varepsilon_t\end{aligned}$$

$$\begin{aligned}\boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} & \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix}\boldsymbol{x}_{t-1}\\ \boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\\ 0 & \phi\end{bmatrix}\boldsymbol{x}_{t-1} & \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi\end{bmatrix}\boldsymbol{x}_{t-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\end{aligned}$$

e. ETS(M,$\mathrm{A_d}$,A)

$$\begin{aligned}y_t &= (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1+\varepsilon_t)\\ \ell_t &= \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t\\ b_t &= \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t\\ s_t &= s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t\end{aligned}$$

$$\begin{aligned}\boldsymbol{x}_t &= \begin{bmatrix}\ell_t & b_t & s_t & s_{t-1} & \dots & s_{t-m+2} & s_{t-m+1}\end{bmatrix}'\\ y_t &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1}(1+\varepsilon_t)\\ \boldsymbol{x}_t &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 0\\ 0 & \phi & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & 0 & \dots & 1 & 0\end{bmatrix}\boldsymbol{x}_{t-1} + \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1}\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}\varepsilon_t\end{aligned}$$

$$\begin{aligned}\boldsymbol{w}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1} & \boldsymbol{r}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1}\\ \boldsymbol{f}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 0\\ 0 & \phi & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & 0 & \dots & 1 & 0\end{bmatrix}\boldsymbol{x}_{t-1} & \boldsymbol{g}(\boldsymbol{x}_{t-1}) &= \begin{bmatrix}1 & \phi & 0 & 0 & \dots & 0 & 1\end{bmatrix}\boldsymbol{x}_{t-1}\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}\end{aligned}$$

Exercise 2.2

a. ETS(A,N,N)

$$\begin{aligned} y_t &= \ell_{t-1} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + \alpha\varepsilon_t \end{aligned}$$

$$\begin{aligned} \hat{y}_{t+1|t} &= \mathrm{E}[y_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t \mid \boldsymbol{x}_t] = \ell_t \\ \hat{y}_{t+2|t} &= \mathrm{E}[y_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1} \mid \boldsymbol{x}_t] = \ell_t \\ &\dots \\ \hat{y}_{t+h|t} &= \mathrm{E}[y_{t+h} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+h-1} + \varepsilon_{t+h} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+h-1} \mid \boldsymbol{x}_t] = \ell_t \end{aligned}$$

$$\begin{aligned} v_{t+1|t} &= \mathrm{var}(y_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\varepsilon_{t+1} \mid \boldsymbol{x}_t) = \sigma^2 \\ v_{t+2|t} &= \mathrm{var}(y_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + \alpha\varepsilon_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) \\ &= (1+\alpha^2)\sigma^2 \\ v_{t+3|t} &= \mathrm{var}(y_{t+3} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+2} + \varepsilon_{t+3} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + \alpha\varepsilon_{t+1} + \alpha\varepsilon_{t+2} + \varepsilon_{t+3} \mid \boldsymbol{x}_t) \\ &= \alpha^2\sigma^2 + \alpha^2\sigma^2 + \sigma^2 = (1+2\alpha^2)\sigma^2 \\ &\dots \\ v_{t+h|t} &= \mathrm{var}(y_{t+h} \mid \boldsymbol{x}_t) = \left[1 + (h-1)\alpha^2\right]\sigma^2 \end{aligned}$$

b. ETS(A,A,N)

$$\begin{aligned} y_t &= \ell_{t-1} + b_{t-1}\varepsilon_t \\ \ell_t &= \ell_{t-1} + b_{t-1}\alpha\varepsilon_t \\ b_t &= b_{t-1} + \beta\varepsilon_t \end{aligned}$$

$$\begin{aligned} \hat{y}_{t+1|t} &= \mathrm{E}[y_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t + b_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t + b_t \mid \boldsymbol{x}_t] = \ell_t + b_t \\ \hat{y}_{t+2|t} &= \mathrm{E}[y_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1} + b_{t+1}\varepsilon_{t+2} \mid \boldsymbol{x}_t] = \ell_t + b_t + b_t = \ell_t + 2b_t \\ &\dots \\ \hat{y}_{t+h|t} &= \mathrm{E}[y_{t+h} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+h-1} + b_{t+h-1} + \varepsilon_{t+h} \mid \boldsymbol{x}_t] = \ell_t + hb_t \end{aligned}$$

$$\begin{aligned} v_{t+1|t} &= \mathrm{var}(y_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + b_t + \varepsilon_{t+1} \mid \boldsymbol{x}_t) = \sigma^2 \\ v_{t+2|t} &= \mathrm{var}(y_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+1} + b_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t + b_t + \alpha\varepsilon_{t+1} + b_t + \beta\varepsilon_{t+1} + \varepsilon_{t+2} \mid \boldsymbol{x}_t) \\ &= (\alpha+\beta)^2\sigma^2 + \sigma^2 = [1 + ((\alpha+\beta)^2]\sigma^2 \\ v_{t+3|t} &= \mathrm{var}(y_{t+3} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+2} + b_{t+2} + \varepsilon_{t+3} \mid \boldsymbol{x}_t) \\ &= \mathrm{var}(\ell_{t+1} + b_{t+1} + \alpha\varepsilon_{t+2} + b_{t+1} + \beta\varepsilon_{t+2} + \varepsilon_{t+3} \\ &= \mathrm{var}[\ell_t + b_t + \alpha\varepsilon_{t+1} + 2(b_t + \beta\varepsilon_{t+1}) + \alpha\varepsilon_{t+2} + \beta\varepsilon_{t+2} + \varepsilon_{t+3}] \\ &= \mathrm{var}[(\alpha+2\beta)\varepsilon_{t+1} + (\alpha+\beta)\varepsilon_{t+2} + \varepsilon_{t+3}] \\ &= [1 + (\alpha+\beta)^2 + (\alpha+2\beta)^2]\sigma^2 = \left[1 + \sum_{j=1}^{2}(\alpha + j\beta^2)\right]\sigma^2 \\ &\dots \\ v_{t+h|t} &= \left[1 + \sum_{j=1}^{h-1}(\alpha + j\beta^2)\right]\sigma^2 \end{aligned}$$

c. ETS(M,N,N)

$$y_t = \ell_{t-1}(1+\varepsilon_t)$$
$$\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$$

$$\begin{aligned}
\hat{y}_{t+1/h} &= \mathrm{E}[y_{t+1} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t(1+\varepsilon_{t+1}) \mid \boldsymbol{x}_t] = \ell_t \\
\hat{y}_{t+2/h} &= \mathrm{E}[y_{t+2} \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_{t+1}(1+\varepsilon_{t+2}) \mid \boldsymbol{x}_t] = \mathrm{E}[\ell_t(1+\varepsilon_{t+1})(1+\varepsilon_{t+2})] = \ell_t \\
&\dots \\
\hat{y}_{t+h/h} &= \mathrm{E}[y_{t+h} \mid \boldsymbol{x}_t] = \ell_t
\end{aligned}$$

$$\begin{aligned}
v_{t+1|t} &= \mathrm{var}(y_{t+1} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_t(1+\varepsilon_{t+1}) \mid \boldsymbol{x}_t) = \ell_t^2\sigma^2 \\
v_{t+2|t} &= \mathrm{var}(y_{t+2} \mid \boldsymbol{x}_t) = \mathrm{var}(\ell_{t+1}(1+\varepsilon_{t+2}) \mid \boldsymbol{x}_t) = \mathrm{var}[\ell_t(1+\alpha\varepsilon_{t+1})(1+\varepsilon_{t+2}) \mid \boldsymbol{x}_t] \\
&= \ell_t^2\mathrm{var}[(1+\alpha\varepsilon_{t+1})(1+\varepsilon_{t+2})] \\
&= \ell_t^2\mathrm{var}[1+\alpha\varepsilon_{t+1}+\varepsilon_{t+2}+\alpha\varepsilon_{t+1}\varepsilon_{t+2}] \\
&= \ell_t^2[\alpha^2\mathrm{var}(\varepsilon_{t+1})+\mathrm{var}(\varepsilon_{t+2})+\alpha^2\mathrm{var}(\varepsilon_{t+1}\varepsilon_{t+2})+2\alpha^2\mathrm{cov}(\varepsilon_{t+1},\varepsilon_{t+1}\varepsilon_{t+2})+2\alpha\mathrm{cov}(\varepsilon_{t+2},\varepsilon_{t+1}\varepsilon_{t+2})] \\
&= \ell_t^2[\alpha^2\sigma^2+\sigma^2+\alpha^2\sigma^2\sigma^2] \\
&= \ell_t^2[(1+\alpha^2\sigma^2)(1+\sigma^2)-1]
\end{aligned}$$

Exercise 2.3

```
> (bonds.ets <- ets(bonds))
ETS(A,Ad,N)

Call:
 ets(y = bonds)

  Smoothing parameters:
    alpha = 0.9999
    beta  = 0.1608
    phi   = 0.8

  Initial states:
    l = 5.5163
    b = 0.2967

  sigma:  0.2394

     AIC     AICc      BIC
256.1641 256.6683 270.3056

> (usnet.ets <- ets(usnetelec))
ETS(M,Md,N)

Call:
 ets(y = usnetelec)

  Smoothing parameters:
    alpha = 0.9999
    beta  = 1e-04
    phi   = 0.9638

  Initial states:
    l = 262.6421
    b = 1.1238
```

```
  sigma:  0.0236

     AIC     AICc      BIC 
628.1943 629.4188 638.2310 

> (ukc.ets <- ets(ukcars))
ETS(A,N,A) 

Call:
 ets(y = ukcars) 

  Smoothing parameters:
    alpha = 0.6267 
    gamma = 2e-04 

  Initial states:
    l = 338.4757 
    s=-0.5313 -45.3246 20.6084 25.2476

  sigma:  25.3264

     AIC     AICc      BIC 
1276.592 1277.385 1292.957 

> (visit.ets <- ets(visitors))
ETS(M,A,M) 

Call:
 ets(y = visitors) 

  Smoothing parameters:
    alpha = 0.6244 
    beta  = 1e-04 
    gamma = 0.1832 

  Initial states:
    l = 86.3534 
    b = 2.0306 
    s=0.942 1.076 1.0515 0.9568 1.3621 1.1157
           1.011 0.8294 0.9336 1.0017 0.8649 0.8554

  sigma:  0.0515

     AIC     AICc      BIC 
2598.193 2600.632 2653.883 
```

Exercise 2.4

```
> forecast(bonds.ets,h=4,level=80)
         Point Forecast    Lo 80    Hi 80
Jun 2004       4.791887 4.485047 5.098727
Jul 2004       4.865425 4.364764 5.366085
Aug 2004       4.924255 4.266113 5.582396
Sep 2004       4.971319 4.172657 5.769980

> forecast(usnet.ets,h=4,level=80)
     Point Forecast    Lo 80    Hi 80
2004       3905.517 3789.206 4026.131
2005       3963.887 3793.446 4133.019
2006       4020.971 3804.828 4224.363
2007       4076.769 3829.443 4314.189

> forecast(ukc.ets,h=4,level=80)
        Point Forecast    Lo 80    Hi 80
2005 Q2       426.8056 394.3485 459.2626
2005 Q3       360.8705 322.5657 399.1753
2005 Q4       405.6569 362.2828 449.0310
2006 Q1       431.4437 383.5363 479.3510

> forecast(visit.ets,h=4,level=80)
         Point Forecast    Lo 80    Hi 80
May 2005       361.1182 337.3021 384.9342
Jun 2005       396.1179 365.3447 426.8912
Jul 2005       494.4950 451.0785 537.9114
Aug 2005       428.0406 386.6065 469.4748
```

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3.6 Exercises

Exercise 3.1

For the ETS(A,N,N) model,

$$y_t = \ell_{t-1} + \varepsilon_t$$
$$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$$

and $\boldsymbol{x}_t = \ell_{t-1}$. Therefore $\boldsymbol{w} = 1$, $\boldsymbol{F} = 1$, $\boldsymbol{g} = \alpha$ and $\boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = 1 - \alpha$.

Forecastability

$$c_j = \boldsymbol{w}'\boldsymbol{D}^{j-1}\boldsymbol{g} = \alpha(1-\alpha)^{j-1}$$
$$\text{and} \qquad a_t = \boldsymbol{w}'\boldsymbol{D}^{t-1}\boldsymbol{x}_0 = (1-\alpha)^{t-1}\ell_0.$$

When $\alpha = 0$, $a_t = \ell_0$ and $c_j = 0$. Therefore $\sum_{j=1}^{\infty} |c_j| = 0$ and $\lim_{t\to\infty} a_t = \ell_0$, and so the process is forecastable.

When $\alpha = 2$, $a_t = (-1)^{t-1}\ell_0$ which does not converge as $t \to \infty$ and so the process is not forecastable.

Stationarity

$$d_t = \boldsymbol{w}'\boldsymbol{F}^{t-1}\boldsymbol{x}_0 = \ell_0$$
$$k_j = \boldsymbol{w}'\boldsymbol{F}^{j-1}\boldsymbol{g} = \alpha, \qquad j \geq 1,$$

and $k_0 = 1$. So for stationarity, we require $\sum_{j=0}^{\infty} |k_j| < \infty$.

When $\alpha = 0$, $k_j = 0$ for all $j \geq 1$ and so the process is stationary.

When $\alpha = 2$, $k_j = 2$ for all $j \geq 1$ and so the process is not stationary.

Exercise 3.2

[This question should have read "forecastable" rather than "stable".]

The local level with drift model is

$$\begin{aligned} y_t &= \ell_{t-1} + b + \varepsilon_t \\ \ell_t &= b + \ell_{t-1} + \alpha\varepsilon_t. \end{aligned}$$

Variable: $z_{1,t} = y_t - bt$: This can be written in state space form as

$$\begin{aligned} z_{1,t} &= \ell_{t-1} + b_{t-1} - u_{t-1} + \varepsilon_t \\ \ell_t &= b_{t-1} + \ell_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} \\ u_t &= u_{t-1} + b_{t-1} \end{aligned}$$

where $b_0 = b$ and $u_0 = b$. Therefore, $\boldsymbol{x}_t = (\ell_t, b_t, u_t)'$, $\boldsymbol{w} = (1, 1, -1)'$, $\boldsymbol{g} = (\alpha, 0, 0)'$,

$$\boldsymbol{F} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = \begin{bmatrix} 1-\alpha & 1-\alpha & \alpha \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{and} \quad \boldsymbol{D}^k = \begin{bmatrix} (1-\alpha)^k & (k-1) + (1-\alpha)^k & 1 - (1-\alpha)^{k+1}] \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}.$$

Therefore $\boldsymbol{w}'\boldsymbol{D}^k = \left[(1-\alpha)^k, (1-\alpha)^k, -(1-\alpha)^{k+1}\right]'$, $a_t = \boldsymbol{w}'\boldsymbol{D}^{t-1}\boldsymbol{x}_0 \to 0$, and $c_j = \boldsymbol{w}'\boldsymbol{D}^{j-1}\boldsymbol{g} = \alpha(1-\alpha)^{j-1}$. If $0 \le \alpha < 2$, then $\sum_{j=1}^{\infty} |c_j|$ is finite and so $z_{1,t}$ is forecastable.

Similarly, $d_t = \boldsymbol{w}'\boldsymbol{F}^{t-1}\boldsymbol{x}_0 = \ell_0$ and $k_j = \boldsymbol{w}'\boldsymbol{F}^{j-1}\boldsymbol{g} = \alpha$ for $j \ge 1$. So $\sum_{j=0}^{\infty} |k_j|$ is infinite and $z_{1,t}$ is not stationary.

Variable: $z_{2,t} = y_t - y_{t-1}$: This can be written in state space form as

$$\begin{aligned} z_{2,t} &= \ell_{t-1} - u_{t-1} + \varepsilon_t \\ \ell_t &= b_{t-1} + \ell_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} \\ u_t &= \ell_{t-1} + \varepsilon_t \end{aligned}$$

where $b_0 = b$. Therefore, $\boldsymbol{x}_t = (\ell_t, b_t, u_t)'$, $\boldsymbol{w} = (1, 0, -1)'$, $\boldsymbol{g} = (\alpha, 0, 1)'$,

$$\boldsymbol{F} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = \begin{bmatrix} 1-\alpha & 1 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$\boldsymbol{D}^k = \begin{bmatrix} (1-\alpha)^k & [1-(1-\alpha)^k]/\alpha & 1-(1-\alpha)^k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore $\boldsymbol{w}'\boldsymbol{D}^k = \left[(1-\alpha)^k,\ [1-(1-\alpha)^k]/\alpha,\ -(1-\alpha)^k\right]'$, $a_t = \boldsymbol{w}'\boldsymbol{D}^{t-1}\boldsymbol{x}_0 \to b_0/\alpha$, and $c_j = \boldsymbol{w}'\boldsymbol{D}^{j-1}\boldsymbol{g} = -(1-\alpha)^j$. So for $0 < \alpha < 2$, $\sum_{j=1}^{\infty} |c_j|$ is finite and $z_{2,t}$ is forecastable.

Similarly, $d_t = \boldsymbol{w}'\boldsymbol{F}^{t-1}\boldsymbol{x}_0 = \ell_0 - u_0$ and $k_j = \boldsymbol{w}'\boldsymbol{F}^{j-1}\boldsymbol{g} = 0$ for $j \ge 2$. So $\sum_{j=0}^{\infty} |k_j|$ is finite and $z_{1,t}$ is stationary.

Exercise 3.3

$$\begin{aligned} y_t &= \ell_{t-1} + \varepsilon_t \\ &= \ell_{t-2} + \alpha\varepsilon_{t-1} + \varepsilon_t \\ &\dots \\ &= \ell_0 + \varepsilon_t + \sum_{j=1}^{t-1} \alpha\varepsilon_{t-j} \end{aligned}$$

Therefore

$$\mathrm{E}(y_t \mid \ell_0) = \ell_0$$

and

$$\mathrm{Var}(y_t \mid \ell_0) = \sigma^2 + (t-1)\alpha^2\sigma^2 = [1 + (t-1)\alpha^2]\sigma^2.$$

[Note that the book has a typo here and replaces σ^2 by ℓ_0^2.]

Exercise 3.4

ETS(A,$\mathrm{A_d}$,N)

$$\begin{aligned} y_t &= \ell_{t-1} + \phi b_{t-1} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t \\ b_t &= \phi b_{t-1} + \beta\varepsilon_t \end{aligned}$$

Therefore $\boldsymbol{x}_t = (\ell_{t-1}, b_{t-1})'$, $\boldsymbol{w}_t = (1, \phi)'$, $\boldsymbol{F} = \begin{bmatrix} 1 & \phi \\ 0 & \phi \end{bmatrix}$, $\boldsymbol{g} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, and

$$\boldsymbol{D} = \boldsymbol{F} - \boldsymbol{g}\boldsymbol{w}' = \begin{bmatrix} 1 & \phi \\ 0 & \phi \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} 1 & \phi \end{bmatrix} = \begin{bmatrix} 1-\alpha & \phi(1-\alpha) \\ -\beta & \phi(1-\beta) \end{bmatrix}.$$

To find the eigenvalues of $\boldsymbol{D}$, we must solve:

$$\det(\boldsymbol{D} - \boldsymbol{I}\lambda) = \det \begin{bmatrix} 1-\alpha-\lambda & \phi(1-\alpha) \\ -\beta & \phi(1-\beta)-\lambda \end{bmatrix} = (1-\alpha-\lambda)(\phi(1-\beta)-\lambda) + \beta\phi(1-\alpha) = 0$$

Therefore

$$\lambda^2 - \lambda[\phi(1-\beta) + (1-\alpha)] + \phi(1-\alpha) = 0,$$

and

$$\lambda = \frac{(1-\alpha) + \phi(1-\beta) \pm \sqrt{[(1-\alpha) + \phi(1-\beta)]^2 - 4\phi(1-\alpha)}}{2}.$$

Solutions to 4.5 Exercises

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Exercise 4.1

Additive Model ETS(A,N,N)

$$\begin{aligned} y_t &= x_{t-1} + \varepsilon_t \\ x_t &= x_{t-1} + \alpha\varepsilon_t \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{V}(y_1 \mid x_0) &= \mathrm{V}(x_0 + \varepsilon_1 \mid x_0) = \sigma_A^2 \\ \mathrm{V}(y_2 \mid x_0) &= \mathrm{V}(x_1 + \varepsilon_2 \mid x_0) = \mathrm{V}(x_0 + \alpha\varepsilon_1 + \varepsilon_2 \mid x_0) \\ &= \alpha^2\sigma_A^2 + \sigma_A^2 = (1+\alpha^2)\sigma_A^2 \\ \mathrm{V}(y_3 \mid x_0) &= \mathrm{V}(x_2 + \varepsilon_3 \mid x_0) = \mathrm{V}(x_1 + \alpha\varepsilon_2 + \varepsilon_3 \mid x_0) \\ &= \mathrm{V}(x_0 + \alpha\varepsilon_1 + \alpha\varepsilon_2 + \varepsilon_3 \mid x_0) \\ &= \alpha^2\sigma_A^2 + \alpha^2\sigma_A^2 + \sigma_A^2 = (1+2\alpha^2)\sigma_A^2 \\ &\vdots \\ \mathrm{V}(y_t \mid x_0) &= \mathrm{V}(x_{t-1} + \varepsilon_t \mid x_0) = [1 + (t-1)\alpha^2]\sigma_A^2 \end{aligned}$$

Multiplicative Model ETS(M,N,N)

$$\begin{aligned} y_t &= x_{t-1}(1 + \varepsilon_t) \\ x_t &= x_{t-1}(1 + \alpha\varepsilon_t) \end{aligned}$$

$$\begin{aligned} \mathrm{V}(y_1 \mid x_0) &= \mathrm{V}\left[x_0(1+\varepsilon_1) \mid x_0\right] \\ &= x_0^2\sigma_M^2 \end{aligned}$$

$$\begin{aligned} \mathrm{V}(y_2 \mid x_0) &= \mathrm{V}\left[x_1(1+\varepsilon_2) \mid x_0\right] \\ &= \mathrm{V}[x_0(1+\alpha\varepsilon_1)(1+\varepsilon_2) \mid x_0] \\ &= x_0^2(\alpha^2\sigma_M^2 + \sigma_M^2 + \alpha^2\sigma_M^2\sigma_M^2) = x_0^2[\alpha^2\sigma_M^2(1+\sigma_M^2) + \sigma_M^2] \\ &= x_0^2[(1+\alpha^2\sigma_M^2)(1+\sigma_M^2) - 1] \end{aligned}$$

$$\begin{aligned} \mathrm{V}(y_3 \mid x_0) &= \mathrm{V}[x_2(1+\varepsilon_3) \mid x_0] \\ &= \mathrm{V}[x_1(1+\alpha\varepsilon_2)(1+\varepsilon_3) \mid x_0] \\ &= \mathrm{V}[x_0(1+\alpha\varepsilon_1)(1+\alpha\varepsilon_2)(1+\varepsilon_3) \mid x_0] \\ &= x_0^2\mathrm{V}(\alpha\varepsilon_1 + \alpha\varepsilon_2 + \varepsilon_3 + \alpha\varepsilon_1\varepsilon_3 + \alpha\varepsilon_2\varepsilon_3 + \alpha^2\varepsilon_1\varepsilon_2 + \alpha^2\varepsilon_1\varepsilon_2\varepsilon_3 + 1) \\ &= x_0^2[\alpha^2\sigma^2 + \alpha^2\sigma_M^2 + \sigma_M^2 + \alpha^2\sigma_M^2\sigma_M^2 + \alpha^2\sigma_M^2\sigma_M^2 + \alpha^4\sigma_M^2\sigma_M^2 + \alpha^4\sigma_M^2\sigma_M^2\sigma_M^2] \\ &= x_0^2[(1+\sigma^2)(1+\alpha^2\sigma_M^2)] \\ &\vdots \\ \mathrm{V}(y_t \mid x_0) &= x_0^2\left[(1+\sigma_M^2)(1+\alpha^2\sigma_M^2)^{t-1} - 1\right] \end{aligned}$$

Exercise 4.2

ETS(A,M,M)

$$
\begin{aligned}
y_t &= \ell_{t-1}b_{t-1}s_{t-m} + \varepsilon_t \\
\ell_t &= \ell_{t-1}b_{t-1} + \alpha\epsilon_t/s_{t-m} \\
b_t &= b_{t-1} + \beta\varepsilon/(s_{t-m}\ell_{t-1}) \\
s_t &= s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1}b_{t-1})
\end{aligned}
$$

In vector form:

$$
\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} (\ell_{t-1}b_{t-1}s_{t-m}) + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t,
$$

or

$$
\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t,
$$

where

$$
\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}
$$

$$
\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b_{t-1}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = 1.
$$

Thus the recursive relationships are given by:

$$
\begin{aligned}
\ell_t &= \ell_{t-1}b_{t-1} - \alpha\ell_{t-1}b_{t-1} + \alpha s_{t-m}/y_t \\
&= (1-\alpha)\ell_{t-1}b_{t-1} + \alpha s_{t-m}/y_t \\
b_t &= b_{t-1} - \beta b_{t-1} + \beta y_t/(s_{t-m}\ell_{t-1}) \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b_{t-1} + \alpha\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha) + \ell_t/\ell_{t-1} + (\beta/\alpha)b_{t-1} + \beta b_{t-1} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b_{t-1} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}) \\
&= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}).
\end{aligned}
$$

ETS(A,Md,M)

$$
\begin{aligned}
y_t &= \ell_{t-1}b_{t-1}^{\phi}s_{t-m} + \varepsilon_t \\
\ell_t &= \ell_{t-1}b_{t-1}^{\phi} + \alpha\epsilon_t/s_{t-m} \\
b_t &= b_{t-1}^{\phi} + \beta\varepsilon_t/(s_{t-m}\ell_{t-1}) \\
s_t &= s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1}b_{t-1}^{\phi})
\end{aligned}
$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b^{\phi}_{t-1} \\ b^{\phi}_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b^{\phi}_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} (\ell_{t-1}b^{\phi}_{t-1}s_{t-m}) + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b^{\phi}_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t,$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t,$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b^{\phi}_{t-1} \\ b^{\phi}_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b^{\phi}_{t-1}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = 1.$$

Thus the recursive relationships are given by:

$$\begin{aligned}
\ell_t &= \ell_{t-1}b^{\phi}_{t-1} - \alpha\ell_{t-1}b^{\phi}_{t-1} + \alpha s_{t-m}/y_t \\
&= (1-\alpha)\ell_{t-1}b^{\phi}_{t-1} + \alpha s_{t-m}/y_t \\
b_t &= b^{\phi}_{t-1} - \beta b^{\phi}_{t-1} + \beta y_t/(s_{t-m}\ell_{t-1}) \\
&= b^{\phi}_{t-1} - \beta b^{\phi}_{t-1} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b^{\phi}_{t-1}\right]/\ell_{t-1} \\
&= b^{\phi}_{t-1} - \beta b^{\phi}_{t-1} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b^{\phi}_{t-1} + \alpha\ell_{t-1}b^{\phi}_{t-1}\right]/\ell_{t-1} \\
&= b^{\phi}_{t-1} - \beta b^{\phi}_{t-1} + (\beta/\alpha) + \ell_t/\ell_{t-1} + (\beta/\alpha)b^{\phi}_{t-1} + \beta b^{\phi}_{t-1} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b^{\phi}_{t-1} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b^{\phi}_{t-1}) \\
&= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b^{\phi}_{t-1}).
\end{aligned}$$

Exercise 4.3

ETS(M,M,M)

$$\begin{aligned}
y_t &= (\ell_{t-1}b_{t-1}s_{t-m})(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1}b_{t-1})(1+\alpha\epsilon_t) \\
b_t &= b_{t-1}(1+\beta\varepsilon_t) \\
s_t &= s_{t-m}(1+\gamma\varepsilon_t)
\end{aligned}$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha(\ell_{t-1}b_{t-1}) \\ \beta b_{t-1} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t,$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha\ell_{t-1}b_{t-1} \\ \beta b_{t-1} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1} \\ b_{t-1} \\ s_{t-m} \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1}b_{t-1}s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = (\ell_{t-1}b_{t-1}s_{t-m}).$$

Thus the recursive relationships are given by:

$$\begin{aligned}
\ell_t &= \ell_{t-1}b_{t-1} - \alpha\ell_{t-1}b_{t-1} + \alpha y_t/s_{t-m} \\
&= (1-\alpha)\ell_{t-1}b_{t-1} + \alpha y_t/s_{t-m} \\
b_t &= b_{t-1} - \beta b_{t-1} + \beta y_t/(s_{t-m}\ell_{t-1}) \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\
&= b_{t-1} - \beta b_{t-1} + (\beta/\alpha)\left[\ell_t - \ell_{t-1}b_{t-1} + \alpha\ell_{t-1}b_{t-1}\right]/\ell_{t-1} \\
&= b_{t-1} - \beta b_{t-1} + \beta/\alpha + \ell_t/\ell_{t-1} + (\beta/\alpha)b_{t-1} + \beta b_{t-1} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha)b_{t-1} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1}) \\
&= (1-\gamma)s_{t-m} + \gamma y_t/(\ell_{t-1}b_{t-1})
\end{aligned}$$

ETS(M,Md,M)

$$\begin{aligned}
y_t &= (\ell_{t-1}b_{t-1}^{\phi}s_{t-m})(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1}b_{t-1}^{\phi})(1+\alpha\epsilon_t) \\
b_t &= b_{t-1}^{\phi}(1+\beta\varepsilon_t) \\
s_t &= s_{t-m}(1+\gamma\varepsilon_t)
\end{aligned}$$

In vector form:

$$\begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} = \begin{bmatrix} \ell_{t-1}b_{t-1}^{\phi} \\ b_{t-1}^{\phi} \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} - \begin{bmatrix} \alpha(\ell_{t-1}b_{t-1}^{\phi}) \\ \beta b_{t-1}^{\phi} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha/s_{t-m} \\ \beta/(s_{t-m}\ell_{t-1}) \\ \gamma/(\ell_{t-1}b_{t-1}^{\phi}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_t$$

or

$$\boldsymbol{x}_t = \left[\boldsymbol{f}(x_{t-1}) - \boldsymbol{g}(x_{t-1})\frac{\boldsymbol{w}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}\right] + \frac{\boldsymbol{g}(x_{t-1})}{\boldsymbol{r}(x_{t-1})}y_t$$

where

$$\boldsymbol{x}_t = \begin{bmatrix} \ell_t \\ b_t \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix} \qquad \boldsymbol{g}(x_{t-1}) = \begin{bmatrix} \alpha(\ell_{t-1}b_{t-1}^{\phi}) \\ \beta b_{t-1}^{\phi} \\ \gamma s_{t-m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{f}(x_{t-1}) = \begin{bmatrix} \ell_{t-1}b_{t-1}^{\phi} \\ b_{t-1}^{\phi} \\ s_t \\ s_{t-1} \\ \vdots \\ s_{t-m+1} \end{bmatrix}$$

$$\boldsymbol{w}(x_{t-1}) = (\ell_{t-1} b_{t-1}^{\phi} s_{t-m}) \qquad \text{and} \qquad \boldsymbol{r}(x_{t-1}) = (\ell_{t-1} b_{t-1}^{\phi} s_{t-m}).$$

Thus the recursive relationships are given by:

$$\begin{aligned}
\ell_t &= \ell_{t-1} b_{t-1}^{\phi} - \alpha \ell_{t-1} b_{t-1}^{\phi} + \alpha y_t / s_{t-m} \\
&= (1-\alpha)\ell_{t-1} b_{t-1}^{\phi} + \alpha y_t / s_{t-m} \\
b_t &= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + \beta y_t / (s_{t-m}\ell_{t-1}) \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha)\left[\ell_t - (1-\alpha)\ell_{t-1} b_{t-1}^{\phi}\right] / \ell_{t-1} \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + (\beta/\alpha)\left[\ell_t - \ell_{t-1} b_{t-1}^{\phi} + \alpha \ell_{t-1} b_{t-1}^{\phi}\right] / \ell_{t-1} \\
&= b_{t-1}^{\phi} - \beta b_{t-1}^{\phi} + \beta/\alpha + \ell_t/\ell_{t-1} + (\beta/\alpha) b_{t-1}^{\phi} + \beta b_{t-1}^{\phi} \\
&= (\beta/\alpha)\ell_t/\ell_{t-1} + (1-\beta/\alpha) b_{t-1}^{\phi} \\
s_t &= s_{t-m} - \gamma s_{t-m} + \gamma y_t / (\ell_{t-1} b_{t-1}^{\phi}) \\
&= (1-\gamma) s_{t-m} + \gamma y_t / (\ell_{t-1} b_{t-1}^{\phi})
\end{aligned}$$

Exercise 4.4

Local trend model

$$\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1+\alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[(\ell_t + b_t)(1+\varepsilon_{t+1}) \mid b_t, \ell_t] = \ell_t + b_t \\
e_{t+1|t} &= y_{t+1} - \hat{y}_{t+1|t} = (\ell_t + b_t)\varepsilon_t \\
\mathrm{V}(e_{t+1|t}) &= (\ell_t + b_t)^2 \sigma^2 \\
\hat{y}_{t+2|t} &= \mathrm{E}[(\ell_{t+1} + b_{t+1})(1+\varepsilon_{t+2}) \mid b_t, \ell_t] \\
&= \mathrm{E}\left\{[(\ell_t + b_t)(1+\alpha\varepsilon_{t+2}) + (b_t + \beta(\ell_t + b_t)\varepsilon_{t+1})(1+\varepsilon_{t+2})] \mid b_t, \ell_t\right\} \\
&= \mathrm{E}\left\{[\ell_t + \alpha\varepsilon_{t+1} + b_t + \alpha b_t \varepsilon_{t+1} + b_t + b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta b_t \varepsilon_{t+1}\varepsilon_{t+2}] \mid b_t, \ell_t\right\} \\
&= \ell_t + b_t + b_t = \ell_t + 2b_t \\
e_{t+2|t} &= y_{t+2} - \hat{y}_{t+2|t} = \alpha\ell_t\varepsilon_{t+1} + \alpha b_t \varepsilon_{t+1} + b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta b_t\varepsilon_{t+1}\varepsilon_{t+2} \\
&= (\alpha\ell_t + \alpha b_t + \beta\ell_t)\varepsilon_{t+1} + b_t\varepsilon_{t+2} + (\beta\ell_t + \beta b_t)\varepsilon_{t+1}\varepsilon_{t+2} \\
\mathrm{V}(e_{t+2|t}) &= (\alpha\ell_t + \alpha b_t + \beta\ell_t)^2\sigma^2 + b_t^2\sigma^2 + (\beta\ell_t + \beta b_t)^2\sigma^2\sigma^2 \\
&= [(\alpha\ell_t + \alpha b_t + \beta\ell_t)^2 + b_t^2 + (\beta\ell_t + \beta b_t)^2\sigma^2]\sigma^2
\end{aligned}$$

Local Level Model with Drift

$$\begin{aligned}
y_t &= (\ell_{t-1} + b)(1+\varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b)(1+\alpha\varepsilon_t)
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[(\ell_t + b)(1+\varepsilon_{t+1}) \mid y_t] = \ell_t + b \\
e_{t+1|t} &= y_{t+1} - \hat{y}_{t+1|t} = (\ell_{t-1} + b)\varepsilon_{t+1} \\
\mathrm{E}[e_{t+1|t}]^2 &= (\ell_{t-1} + b)^2\sigma^2 \\
\hat{y}_{t+2|t} &= \mathrm{E}[(\ell_{t+1} + b)(1+\varepsilon_{t+2}) \mid y_t] = \mathrm{E}\left\{[(\ell_t + b)(1+\alpha\varepsilon_{t+1}) + b](1+\varepsilon_{t+2}) \mid y_t\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left\{[(\ell_t + b) + (\ell_t + b)\alpha\varepsilon_{t+1} + b](1 + \varepsilon_{t+2}) \mid y_t\right\} \\
&= \mathrm{E}[(\ell_t + 2b) + (\ell_t + b)\alpha\varepsilon_{t+1} + (\ell_t + 2b)\varepsilon_{t+2} + (\ell_t + b)\alpha\varepsilon_{t+1}\varepsilon_{t+2}] = \ell_t + 2b \\
e_{t+2|t} &= y_{t+2} - \hat{y}_{t+2|t} \\
&= \alpha(\ell_t + b)\varepsilon_{t+1} + (\ell_t + 2b)\varepsilon_{t+2} + (\ell_t + b)\alpha\varepsilon_{t+1}\varepsilon_{t+2} \\
\mathrm{E}[e_{t+2|t}]^2 &= \alpha^2(\ell_t + b)^2\sigma^2 + (\ell_t + 2b)^2\sigma^2 + (\ell_t + b)^2\alpha^2\sigma^2\sigma^2 \\
&= \alpha^2(\ell_t + b)^2\sigma^2(1 + \sigma^2) + (\ell_t + 2b)^2\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+3|t} &= \mathrm{E}[(\ell_{t+2} + b)(1 + \varepsilon_{t+3}) \mid y_t] \\
&= \mathrm{E}\left\{[(\ell_{t+1} + b)(1 + \alpha\varepsilon_{t+2}) + b](1 + \varepsilon_{t+3}) \mid y_t\right\} \\
&= E\left\{[[(\ell_t + b)(1 + \alpha\varepsilon_{t+1}) + b](1 + \alpha\varepsilon_{t+2}) + b](1 + \varepsilon_{t+3}) \mid y_t\right\} \\
&= \ell_t + b + b + b = \ell + 3b \\
e_{t+3|t} &= y_{t+3} - \hat{y}_{t+3|t} \\
&= (\ell_t + b)\alpha\varepsilon_{t+1} + (\ell_t + 2b)\alpha\varepsilon_{t+2} + (\ell_t + b)\alpha^2\varepsilon_{t+1}\varepsilon_{t+2} + (\ell_t + 3b)\varepsilon_{t+3} \\
&+ (\ell_t + b)\alpha\varepsilon_{t+1}\varepsilon_{t+3} + (\ell_t + 2b)\alpha\varepsilon_{t+2}\varepsilon_{t+3} + (\ell_t + b)^2\varepsilon_{t+1}\varepsilon_{t+2}\varepsilon_{t+3} \\
\mathrm{E}[e_{t+3|t}]^2 &= (\ell_t + b)^2\alpha^2\sigma^2 + (\ell_t + 2b)^2\alpha^2\sigma^2 + (\ell_t + b)^2\alpha^4\sigma^2\sigma^2 + (\ell_t + 3b)^2\sigma^2 \\
&+ (\ell_t + b)^2\alpha^2\sigma^2\sigma^2 + (\ell_t + 2b)^2\alpha^2\sigma^2\sigma^2 + (\ell_t + b)^2\alpha^4\sigma^2\sigma^2\sigma^2 \\
&= (\ell_t + b)^2\alpha^2\sigma^2[1 + 2\alpha^2\sigma^2 + \alpha^2\sigma^4] + (\ell_t + 2b)^2\sigma^2\alpha^2(1 + \sigma^2) + (\ell_t + 3b)^2\sigma^2
\end{aligned}
$$

Exercise 4.5

Damped trend model

$$
\begin{aligned}
y_t &= (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + \phi b_{t-1}) + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+1|t} &= \mathrm{E}[(\ell_t + \phi b_t)(1 + \varepsilon_{t+1}) \mid y_t] = \ell_t + \phi b_t \\
e_{t+1|t} &= y_{t+1} - \hat{y}_{t+1|t} = (\ell_t + \phi b_t)\varepsilon_t \\
\mathrm{V}(e_{t+1|t}) &= (\ell_t + \phi b_t)^2\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+2|t} &= \mathrm{E}[(\ell_{t+1} + \phi b_{t+1})(1 + \varepsilon_{t+2}) \mid y_t] \\
&= \mathrm{E}\left\{[(\ell_t + \phi b_t)(1 + \alpha\varepsilon_{t+2}) + \phi(\phi b_t + \beta(\ell_t + \phi b_t)\varepsilon_{t+1})(1 + \varepsilon_{t+2})] \mid y_t\right\} \\
&= \mathrm{E}\left\{\left[\ell_t + \alpha\varepsilon_{t+1} + \phi b_t + \phi\alpha b_t\varepsilon_{t+1} + \phi^2 b_t + \phi^2 b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta\phi b_t\varepsilon_{t+1}\varepsilon_{t+2}\right] \mid y_t\right\} \\
&= \ell_t + \phi b_t + \phi^2 b_t = \ell_t + \phi b(1 + \phi) \\
e_{t+2|t} &= y_{t+2} - \hat{y}_{t+2|t} = \alpha\ell_t\varepsilon_{t+1} + \phi\alpha b_t\varepsilon_{t+1} + \phi^2 b_t\varepsilon_{t+2} + \beta\ell_t\varepsilon_{t+1} + \beta\ell_t\varepsilon_{t+1}\varepsilon_{t+2} + \beta b_t\varepsilon_{t+1}\varepsilon_{t+2} = \\
&= (\alpha\ell_t + \phi\alpha b_t + \beta\ell_t)\varepsilon_{t+1} + \phi^2 b_t\varepsilon_{t+2} + (\beta\ell_t + \beta\phi b_t)\varepsilon_{t+1}\varepsilon_{t+2} \\
\mathrm{V}(e_{t+2|t}) &= (\alpha\ell_t + \phi\alpha b_t + \beta\ell_t)^2\sigma^2 + \phi^4 b_t^2\sigma^2 + (\beta\ell_t + \beta\phi b_t)^2\sigma^2\sigma^2 \\
&= [(\alpha\ell_t + \phi\alpha b_t + \beta\ell_t)^2 + \phi^4 b_t^2 + (\beta\ell_t + \beta\phi b_t)^2\sigma^2]\sigma^2
\end{aligned}
$$

Exercise 4.6

The ETS(M,A,N) model is given by

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\epsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t
\end{aligned}
$$

$$
\boldsymbol{x}_t = [\ell_t, b_t]', \qquad w(\boldsymbol{x}_{t-1}) = \ell_{t-1} + b_{t-1}, \qquad r(\boldsymbol{x}_{t-1}) = \ell_{t-1} + b_{t-1},
$$

$\boldsymbol{f}(\boldsymbol{x}_{t-1}) = [\ell_{t-1} + b_{t-1}, b_{t-1}]'$, $\quad \boldsymbol{g}(\boldsymbol{x}_{t-1}) = [\alpha(\ell_{t-1} + b_{t-1}), \beta(\ell_{t-1} + b_{t-1})]'$, $\quad$ and

$$\begin{aligned}
\boldsymbol{D} &= \boldsymbol{f}(\boldsymbol{x}_t) - \boldsymbol{g}(\boldsymbol{x}_t) w(\boldsymbol{x}_t)/r(\boldsymbol{x}_t) \\
&= \begin{bmatrix} \ell_t + b_t \\ b_t \end{bmatrix} - \begin{bmatrix} \alpha(\ell_t + b_t) \\ \beta(\ell_t + b_t) \end{bmatrix} \\
&= \begin{bmatrix} (\ell_t + b_t) - \alpha(\ell_t + b_t) \\ b_t - \beta(\ell_t + b_t) \end{bmatrix} \\
&= \begin{bmatrix} (1-\alpha)(\ell_t + b_t) \\ -\beta\ell_t + (1-\beta)b_t \end{bmatrix} \\
&= \begin{bmatrix} (1-\alpha) & (1-\alpha) \\ -\beta & (1-\beta) \end{bmatrix} \begin{bmatrix} \ell_{t-1} \\ b_{t-1} \end{bmatrix}
\end{aligned}$$

Eigenvalues

$$\begin{aligned}
\boldsymbol{D} - \boldsymbol{I}\lambda &= \begin{bmatrix} (1-\alpha) & (1-\alpha) \\ -\beta & (1-\beta) \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha) - \lambda & (1-\alpha) \\ -\beta & (1-\beta) - \lambda \end{bmatrix} \\
&= [(1-\alpha) - \lambda][1 - \beta - \lambda] + \beta(1-\alpha) = 0
\end{aligned}$$

$$\lambda = \frac{1}{2}\left(2 - \alpha - \beta \pm \sqrt{(\alpha+\beta)^2 + 4\beta)}\right)$$

So $|\lambda| < 1$ iff $\alpha > 0$ and $0 < \beta < 4 - 2\alpha$.

Exercise 4.7

```
require(expsmooth)
plot(djiclose)
x <- window(djiclose[,"close"],start=1980)
fit <- forecast(ets(x,"MAN"),h=50)
fit2 <- rwf(x,drift=TRUE,h=50)
par(mfrow=c(2,2))
plot(fit)
plot(fit2)
plot(residuals(fit))
plot(residuals(fit2))
```

The plot shows that the random walk with drift model has much smaller forecast intervals. It has underestimated the forecast variance because of the heterogeneous residual.